Lecture 12: Heat Equation Part 2

Integral Solution Formula

- Consider

$$(A)\ \begin{cases} \partial u/\partial t - \Delta u = 0 & [0,\infty)_t \times \mathbb{R}^n \\ u(0,x) = g(x) & \mathbb{R}^n \end{cases}$$

- As in lecture 11, define $H_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$

  Such that $\int_{\mathbb{R}^n} H_t(x)\,dx = 1$

  Directly differentiating shows $\partial/\partial t\, H_t - \Delta H_t = 0$

  Weirdly, $\lim\limits_{t\to 0} H_t(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$

$\boxed{\text{Th}^{\text{m}}\ 6.2}$ For a bounded function $g \in C^0(\mathbb{R}^n)$,

$$u(t,x) = H_t * g(x) := (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\,dy$$

Solves (A), and $u \in C^2$

$\boxed{\text{Pf}}$ We would like to differentiate through the integral, but this takes some care because the domain is infinite.

To justify this, we estimate the partials of H_t by expressions of the form $c_1(t,x)\, e^{-c_2(t,x)|y|^2}$ for c_1, c_2 continuous. However, we need to build theory to make such estimations valid that are beyond this course (mainly distribution theory).

Assuming we can, however,

$$(\partial_t - \Delta)\cdot u(t,x) = \cancel{-\tfrac{n}{2}(4\pi t)^{-n/2-1}(4\pi)\int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\,dy}$$

$$= \int_{\mathbb{R}^n} \left[(\partial_t - \Delta_x) H_t(x-y)\right] g(y)\,dy = \int_{\mathbb{R}^n} 0 \cdot g(y)\,dy = 0$$

To check the initial condition, fix $x \in \mathbb{R}^n$ and set $w = \frac{y-x}{\sqrt{t}}$

so

$$u(t,x) = (4\pi \cancel{t})^{-n/2} \int_{\mathbb{R}^n} e^{-|w|^2} g(x + t^{1/2} w)\,dw$$

↳ t eliminated by change-of-variables

$$= \int_{\mathbb{R}^n} H_1(w)\, g(x + t^{1/2} w)\,dw$$

Notice $g(x) = \int_{\mathbb{R}^n} H_1(w)\, g(x)\,dw$

So $\quad u(t,x) - g(x) = \int_{\mathbb{R}^n} \mathcal{H}_1(w)\left[g(x+t^{1/2}w) - g(x)\right]dw$

Our idea is that as $t \to 0$, $g(x+t^{1/2}w) - g(x) \to 0$. We make this rigorous. Pick $\varepsilon > 0$.

First, pick $R > 0$ such that $\int_{B(0,R)} \mathcal{H}_1(w)\,dw > 1 - \varepsilon$.

Next, g is bounded, so $|g(x)| \leq M$ for some $M > 0$. Since g is continuous & $\overline{B(0,R)}$ is compact, g is absolutely continuous and $\exists$ $\delta > 0$ such that for $|x-y| < \delta$, $|g(x) - g(y)| < \varepsilon$.

Further, pick $\gamma > 0$ so $\gamma < \delta R^2/w$. Then,

for $t < \gamma$, $|t^{1/2}w| < \delta$ for $w \in \overline{B(0,R)}$ and

$$\left|\int_{\mathbb{R}^n} \mathcal{H}_1(w)\left[g(x+t^{1/2}w) - g(x)\right]dw\right| \leq$$

$$\left|\int_{B(0,R)} \mathcal{H}_1(w)\left[g(x+t^{1/2}w) - g(x)\right]dw\right| + \left|\int_{\mathbb{R}^n \setminus B(0,R)} \mathcal{H}_1(w)\left[g(x+t^{1/2}w) - g(x)\right]dw\right|$$

$$\leq \int_{B(0,R)} \mathcal{H}_1(w)\,\varepsilon\,dw + \int_{\mathbb{R}^n \setminus B(0,R)} \mathcal{H}_1(w)(2M)\,dw$$

$$\leq (1)(\varepsilon) + 2M\varepsilon = \varepsilon(2M+1)$$

Hence, $\lim_{t \to 0} u(t,x) = g(x)$. $\square$

$\boxed{\text{Th}^{\text{m}}\ 6.3}$ Under the assumption that $u(t,\cdot)$ is bounded on $[0,\tau] \times \mathbb{R}^n$ for each $\tau > 0$, the solution to the heat equation given above is unique.

$\leadsto$ Proved in Ch. 9 by maximum principles.

$\boxed{\text{Th}^{\text{m}}\ 6.4}$ Suppose that u is a bounded solution of the heat equation (A) for bounded initial conditions $g \in C^0(\mathbb{R}^n)$. Then, $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$

• Remarks:

1.) $H_t(x) > 0$ for all $t>0$, $x \in \mathbb{R}^n$. If $g(x) \geq 0$, then $u(t,x) > 0$ at all $x \in \mathbb{R}^n$ for $t>0$. This is called infinite propagation speed.

2.) Theorem 6.3 may be strengthened to $u(t,\cdot)$ merely having sub-exponential growth.

3.) Theorem 6.4 may be proven by distributional methods or Fourier-analytic methods.

<u>The Inhomogeneous Problem</u>

• We apply Duhamel's Principle as in the wave equation case.

Consider (H2) $\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f \\ u(0,x) = 0 \end{cases}$

For $s \geq 0$, let $\eta_s(t,x)$ be the solution of $\frac{\partial \eta_s}{\partial t} - \Delta \eta_s = 0$ in time $t \geq s$. for $\eta_s(t,x)|_{t=s} = f(s,x)$.

We claim that
$$u(t,x) = \int_0^t \eta_s(t,x)\,ds$$
is the solution to (H2).

The formula for η_s would then give
$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} H_{t-s}(x-y) f(s,y)\, d^n y\, ds. \qquad (S)$$

$\boxed{\text{Th}^{\text{m}}\ 6.5}$ Assuming $f \in C_c^2([0,\infty) \times \mathbb{R}^n)$, (S) gives a classical solution to (H2).

$\boxed{\text{Pf}}$ Notice that $u(t,x) = \int_0^t \int_{\mathbb{R}^n} H_s(y) f(t-s, x-y)\, dy\, ds$ shows $u \in C^2$. Since $H_s(y)$ is smooth near $s=t$ and f is compactly supported, we can differentiate under the integral

$$\frac{\partial u}{\partial t}(t,x) = \int_0^t \int_{\mathbb{R}^n} \mathcal{H}_s(y) \,\partial f/\partial t\,(t-s, x-y)\,dy\,ds$$

$$+ \int_{\mathbb{R}^n} \mathcal{H}_t(y)\, \partial f/\partial t\,(0, x-y)\,dy$$

and

$$\Delta u(t,x) = \int_0^t \int_{\mathbb{R}^n} \mathcal{H}_s(y) \Delta_x f(t-s, x-y)\,dy\,ds$$

Our goal is to carefully integrate-by-parts and use that $\mathcal{H}_s$ solves the heat equation.

- To deal with the singularity, we split at $s=\varepsilon$:

$$\int_\varepsilon^t \int_{\mathbb{R}^n} \mathcal{H}_s(y)\,\partial f/\partial t\,(t-s, x-y)\,dy\,ds =$$

$$-\int_\varepsilon^t \int_{\mathbb{R}^n} \mathcal{H}_s(y)\,\partial f/\partial s\,(t-s, x-y)\,dy\,ds$$

$$\cancel{= -\int_\varepsilon^t \int_{\mathbb{R}^n} \mathcal{H}_s(y)\,\partial f/\partial s\,(t-s, x-y)}$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \frac{\partial}{\partial s}\mathcal{H}_s(y)\, f(t-s, x-y)\,dy\,ds$$

$$- \int_{\mathbb{R}^n} \mathcal{H}_t(y) f(0, x-y)\,dy + \int_{\mathbb{R}^n} \mathcal{H}_\varepsilon(y) f(t-\varepsilon, x-y)\,dy$$

our other term has

$$\int_\varepsilon^t \int_{\mathbb{R}^n} \mathcal{H}_s(y) \Delta_x f(t-s, x-y)\,dy\,ds$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Delta_y \mathcal{H}_s(y)\, f(t-s, x-y)\,dy\,ds$$

s.t.

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = \int_{\mathbb{R}^n} \mathcal{H}_\varepsilon(y)\, f(t-\varepsilon, x-y)\,dy \quad \Big]\ A$$

$$+ \int_0^\varepsilon \int_{\mathbb{R}^n} \mathcal{H}_s(y)\left(\frac{\partial}{\partial t} - \Delta_x\right) f(t-s, x-y)\,dy\,ds \quad \Big]\ B$$

$$+ \int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{\left(\frac{\partial}{\partial s} - \Delta_y\right)\mathcal{H}_s(y)}_{0}\, f(t-s, x-y)\,dy\,ds$$

. Since $H_s > 0$, B may be estimated by

$$\left| \int_0^{\varepsilon} \int_{\mathbb{R}^n} H_s(y) \left(\frac{\partial}{\partial t} - \Delta_x\right) f(t-s, x-y)\, dy\, ds \right| \leq c \int_0^{\varepsilon} \int_{\mathbb{R}^n} H_s(y)\, dy\, ds \leq c\varepsilon$$

for $c = \max \left\{ \left(\frac{\partial}{\partial t} - \Delta_x\right) f \right\}$ that exists b/c $f \in C_c^2$.
and since $\int_{\mathbb{R}^n} H_s(y)\, dy = 1$.

Thus,

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} H_\varepsilon(y) f(t-\varepsilon, x-y)\, dy = f(t,x)$$

as in the previous calculation.